

Donaldson, The Ding functional, Berndtsson (IHES 7/2014) Convexity and Moment Maps

Formal diff. geom. aspects of KE, csck metrics
scalar curv.

$$1) \quad L \rightarrow V \\ h \rightarrow \omega_h > 0 \quad S(\omega_h) = \text{const.} \quad 4^{\text{th}} \text{ orden PDE.}$$

$\text{Trace}(\rho)$ Ricci curv.

$$2) \quad L = K_V^{-1} \quad \text{Fano} \quad \text{KE} \quad \rho(\omega_h) = \omega_h$$

$$h \mapsto \Omega_h \quad \text{volume form} \quad \boxed{\Omega_h = \omega_h^n / n!} \quad \begin{matrix} \uparrow \\ \downarrow \\ 2^{\text{nd}} \text{ order PDE} \end{matrix}$$

$$\text{In case (2), } \bar{\partial}^* \rho = \partial S$$

$$S = \text{const} \iff \rho \text{ harmonic}$$

$$[\rho] = [\omega] \quad \xrightarrow{\text{(uniqueness)}} \quad \rho = \omega.$$

$$\text{Ding functional} \quad D_0(h) \stackrel{\cong}{=} \log(\int_V \Omega_h)$$

$$h = e^\varphi h_0 \quad = \log(\int_V e^{-\varphi} \Omega_{h_0})$$

Berndtsson D_0 convex.

WANT TO FIT THESE IDEAS INTO A
"STANDARD FRAMEWORK"

STANDARD FRAMEWORK.

G acts on (Z, σ) Kähler holom. isometry

$\mu: Z \rightarrow \sigma^*$ moment map

$\sigma \xrightarrow{\rho^*} TZ$ (i.e. $d\mu = \text{adj. of } \rho^*$ (via $\omega: T \leftrightarrow T^*$))

$\sigma^* \xleftarrow{d\mu} G^c$ acts holomorphically on Z

Compare complex quotient Z/G^c
with symplectic quotient $\mu^{-1}(0)/G$.

Fix a G^c -orbit, $G^c z_0 \subset Z$

$$F : G^c \rightarrow \mathbb{R}$$

w/ $\delta_z z = \delta g \cdot z_0$ s.t. $\delta F = \langle i\delta g, \mu(z) \rangle \leftarrow G\text{-inv.}$

induces $F : G^c/G \rightarrow \mathbb{R}$ convex (along geodesics)

$$\min \longleftrightarrow \mu(z) = 0$$

$U \simeq \mathbb{R}^{2n}$, ω sympl. form.

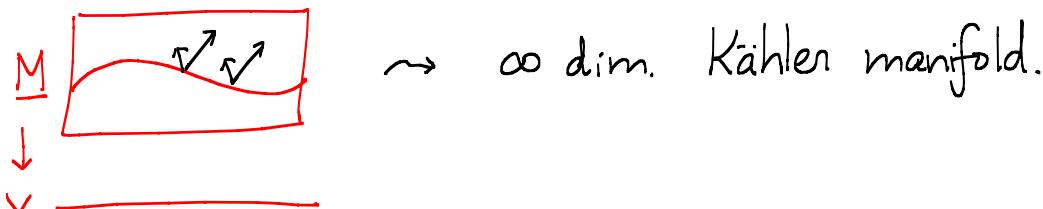
$M := \{ \text{cx. str. on } U \text{ compatible with } \omega \} = \frac{\text{Sp}(2n, \mathbb{R})}{\text{U}(n)}$

has an $\text{Sp}(2n, \mathbb{R})$ -inv. Kähler structure

- $n=1$, $\text{Sp}(2, \mathbb{R})/\text{U}(1) =$ hyperbolic plane.

$$\begin{array}{ccc} L & \xrightarrow{\quad} & X, \omega \\ \text{conn.} \quad \triangledown \\ \text{curv.} - i\omega & & \text{cpt. symplectic} \\ & & \end{array} \quad \alpha \in X \xrightarrow{\sim} M_\alpha \xrightarrow{\text{compat. cx. str. on } \overline{\text{ta}}X} \text{bundle } M \xrightarrow{\sim} \underline{M} \rightarrow X$$

$\Gamma(\underline{M})$ compat. almost complex structures



$$\mathcal{G} := \text{Aut}(X, L, \omega) \xrightarrow{\text{not preserving } X} \text{SDiff}(X, \omega)$$

$$\text{Lie}_{\mathcal{G}} = C^\infty(X)$$

$$\mu_0(J) = S(J) \quad \text{scalar curv.}$$

$$\mu(J) = S(J) - \hat{S} \leftarrow \text{average / topo. inv.}$$

$$\mu^{-1}(0) = \{ \text{csc metrics} \}$$

Restrict to: $Z_{\text{int}} \subset Z = \Gamma(\underline{M})$
integrable alm. cx. str.

In the Fano case, we can study a different Kähler metric on Z_{int} .

Review $M^{\text{open}} \subset \{\text{Lagr. subsp. of } U \otimes \mathbb{C}\} \subset \mathbb{P}(\wedge^n(U \otimes \mathbb{C}))$
 \mathbb{C}^n -bdl. $\hat{M} \rightarrow M$ equiv. to $a dz_1 \wedge \dots \wedge dz_n = da$

$$\langle \alpha, \beta \rangle = c_n \alpha \wedge \bar{\beta}, \quad \text{cn st. } \langle d\alpha, d\beta \rangle = \frac{\omega^n}{n!}$$

$$\alpha \in \hat{M} \quad T\hat{M} \supset \mathbb{C}\alpha$$

Fact: $\langle \cdot, \cdot \rangle$ negative def. on ortho. complement
of α in $(T\hat{M})_\alpha$

$$\beta \in T\hat{M} \quad \langle \beta, \beta \rangle - \frac{|\langle \alpha, \beta \rangle|^2}{\langle \alpha, \alpha \rangle} \leq 0$$

$$"\leq" \text{ iff } \beta \in \mathbb{C}\alpha$$

\mapsto metric on M

Assume $c_1(X) = c_1(L)$

$$x \in X \quad \hat{M}_x \subset \wedge^n T_x^* \otimes L$$

$$\mapsto \text{bundle} \quad \underline{\hat{M}} \longrightarrow X$$

$c_1(X) = c_1(L) \Rightarrow$ existence of global section
 $\alpha \in \Omega^{n,0}(X, L)$

Lemma 1. $\hat{Z}_{\text{int}} := \{\alpha \in \Gamma(\hat{M}) : d\nabla \alpha = 0\}$

\mathbb{C}^n -bundle over Z_{int} . (Pf i.e. integrability).

$\alpha \in Z_{\text{int}}$, $T_\alpha = \text{tangent space to } \hat{Z}_{\text{int}} \text{ at } \alpha$

$= \{\beta : d\nabla \beta = 0 \text{ satisfy alg. condition at each pt.}\}$

$$\langle\langle \beta_1, \beta_2 \rangle\rangle \triangleq \int_X \langle \beta_1, \beta_2 \rangle$$

* Lemma 2 $\Leftarrow \Rightarrow$ neg. def. on ortho. comp of α in T_α

From lemma 2, get induced neg. def.
Kähler structure on Z^{int}

Action of Lie algebra $_{H\in} C^\infty(X)$ on $_{\alpha\in} \Omega^n(L)$,

$$R_\alpha(H) = (v_H \lrcorner \underbrace{d\alpha}_{=0}) + d(v_H \lrcorner \alpha) + iH\alpha$$

Moment map $\alpha \mapsto (H \mapsto \langle R_\alpha H, \alpha \rangle)$

$$= \int \langle d(v_H \lrcorner \alpha) + iH\alpha, \alpha \rangle \quad (\because d\alpha = 0)$$

$$= \int H \langle \alpha, \alpha \rangle.$$

$$\mu_0(\alpha) = \langle \alpha, \alpha \rangle \quad \text{vol. form (i.e. Ding fct.)}$$